Numerical Methods for Hyperbolic and Kinetic Equations

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Phenomena characterized by conservation (or balance laws) of physical quantities are modelled by hyperbolic and kinetic equations. Thus these equations appear in several fields of applied mathematics, such as fluid dynamics, rarefied gas dynamics, behaviour of semiconductor devices, magneto fluid dynamics, astrophysics, traffic flows, to mention just a few.

The numerical treatment of such equations however is difficult, because solutions can develop singularities in a finite time, or become very steep, and, in many cases, uniqueness of the solution can be obtained only imposing further constraints. As a consequence of these difficulties, numerical solutions can develop spurious oscillations, and even converge to the wrong solution. Moreover, kinetic equations can be solved, at present, through schemes characterized by a very high computational complexity. For these reasons, the numerical integration of conservation laws has attracted a lot of attention from the academic community.

First order schemes for conservation laws were developed in the '50s, by the pioneering work of Godunov and Lax. These schemes are robust, but, because of their low accuracy, are unable to resolve solutions with structure. In the '80s, the theory and construction of non-oscillatory second order schemes underwent a fast and impressive development, thanks to the work of Harten, Osher, Van Leer, Roe, and many others. In the late '80s, Harten, Osher and coworkers introduced ENO schemes, which paved the way to the construction of higher order schemes. The schemes developed in this period of time, however are expensive, difficult to implement and highly problem dependent.

The most recent developments in this field can be grouped together under the idea of constructing more flexible and efficient schemes. In this line of thought, we mention WENO schemes, by Osher, Shu and coworkers, central schemes, initially proposed by Nessyahu and Tadmor, and relaxation schemes, first proposed by Jin and Xin. These schemes do need neither costly exact or approximate Riemann solvers, nor projection along characteristic directions. Although mostly they were initially developed and tested for the classical problems of gas dynamics, they have been applied succesfully to a wide field of problems arising form applied mathematics.

These developments contribute to bridge the gap between the sophistication and complexity of research codes, on one side, and, on the other side, the robustness and flexibility asked for in commercial codes.

In this minisymposium, further developments and applications are discussed. The first paper, *Central schemes for balance laws* by G. Russo is a comprehensive review of high order central schemes applied to balance laws. The second paper, *A Naive implementation of ACM in nonoscillatory central difference schemes for 2D Euler equations*, by K.A. Lie proposes a simple and effective technique to improve the resolution of contact discontinuities. The motivation for this work derives from the fact that contact discontinuities in general are poorly resolved by finite difference schemes. The third paper, *Numerical solution of the non-homogeneous Fokker-Planck-Landau equation*, by F. Filbet and L. Pareschi describes a coupling between spectral methods for the collision part and non-oscillatory reconstruction for the convective part, to efficiently solve a kinetic model. Besides its low computational cost, this scheme is interesting in the present framework, because it illustrates how techniques initially developed for hyperbolic equations can be succesfully applied to more complex problems, such as kinetic models.

Several models are given by interacting phenomena, characterized by different scales. Often the macroscopic description is a good approximation to the solution. Hence the need to obtain numerical schemes which can produce a reliable macroscopic description, without resolving the underlying microscales. This issue ia addressed in *Multiscale hyperbolic equations: numerical approximation and applications*, by G. Naldi.

Finally an approach to the solution of problems with moving boundaries can be found in *Singularity and numerical analysis of a singular moving boundary hyperbolic problem*, by R. Fazio. This is an example of the new developments in a promising field of research for conservation laws: the optimization of resources through the construction of adaptive and moving grids for the integration of conservation laws.

Central Schemes for Balance Laws

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Abstract. A brief review is given of shock capturing central schemes for the numerical solution of hyperbolic systems of balance laws. It is shown how to construct high order schemes for conservation laws on a staggered mesh, by using Central Weighted Essentially Non-Oscillatory reconstruction, and how to construct second order central schemes for systems with stiff source which are accurate in the stiff limit. The development of higher order schemes for systems with stiff source is also discussed.

1 Central Schemes for Conservation Laws

The purpose of this talk is to give a brief review of shock capturing central schemes for the numerical solution of systems of conservation laws, and for the treatment of hyperbolic systems with source.

We consider a system of balance laws of the form

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = R(u(x,t)), \tag{1}$$

$u \in \mathbb{R}^m$; $f, R : \mathbb{R}^m \to \mathbb{R}^m$, $A = \nabla_u f(u)$ has real eigenvalues and a basis of eigenvectors. If $R \equiv 0$ the system is said to be of conservation laws. Systems of this form appear in many physical systems, such as gas dynamics, magnetohydrodynamics, shallow water equations, hydrodynamical models of semiconductors, just to mention some typical examples. In the first two cases it is $R \equiv 0$, while in the latter two there is a source term on the right hand side.

Finite volume schemes in conservation form can be divided into two broad families: upwind schemes and central schemes.

The prototype of upwind schemes is Godunov scheme. Space and time are discretized into cells. At each time step t_n, the solution $u(x, t_n)$ is approximated by a piecewise constant function, the value in each cell representing an approximation of the cell average of $u(., t_n)$. The cell average at the new time step is obtained by integrating the conservation law (1) on the border of the cell in space time (see Fig. 1, left). For linear systems Godunov methods is equivalent to the classical first order upwind scheme. Such method is first order accurate, it is Total Variation Diminishing, and it satisfies a discrete entropy inequality.

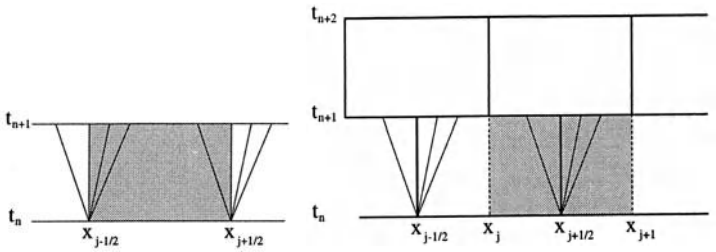

Fig. 1. Integration of the equation over non-staggered cell j (Godunov schemes), and staggered cell $j + 1/2$ (staggered central schemes)

High order versions of Godunov scheme are given by the popular ENO (Essentially Non Oscillatory) and WENO (Weighted ENO) schemes. They are based on a piecewise polynomial reconstruction of the function in each space cell.

The idea behind ENO is the following: choose a degree p, and reconstruct the function to order $p + 1$ with the least oscillatory polynomial of degree p. This can be effectively obtained by a recursive algorithm, that has been widely used in the applications. An very clear discussion on ENO and WENO scheme is given in [11].

An evident disadvantage of Godunov method is that it requires the knowledge of the solution to the Riemann problem (or at least a numerical approximation of it). In several relevant cases, such solution is know (Riemann solver). However, there are several other cases where there is no simple expression of the eigenvalues and eigenvectors of the characteristic matrix A of the system. In such cases it would be desirable to have a scheme that does not rely on the solution to the Riemann problem.

The family of central schemes is obtained by integrating equation (1) on a staggered grid in space time (see Fig. 1, right), obtaining

$$\Delta x\, \bar{u}^{n+1}_{j+1/2} = \int_{x_j}^{x_{j+1}} u(x, t_n)\, dx - \int_{t_n}^{t^{n+1}} (f(u(x_{j+1}, t)) - f(u(x_j, t)))\, dt + \int_{x_j}^{x_{j+1}} \int_{t_n}^{t^{n+1}} R(u(x,t))\, dx\, dt. \tag{2}$$

where $\bar{u}^{n+1}_{j+1/2}$ denotes the cell average on the staggered cell $j + 1/2$ at time t_{n+1}. In order to convert the above (exact) expression into a numerical scheme one has to specify how to perform the piecewise polynomial approximation of the function $u(x, t_n)$ from cell averages, and how to approximate the time integral of the flux.

Piecewise linear approximation of the function, in addition with midpoint rule for the integral of the flux, and Explicit Euler scheme for the predictor value gives the well-known Nessyahu-Tadmor scheme, straightforwardly ex-

tended to systems with non-stiff source,

$$
\begin{aligned}
u_j^{n+1/2} &= u_j^n - \frac{\lambda}{2} f_j' + \frac{\Delta t}{2} R(u_j^n), \\
u_{j+1/2}^{n+1} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') - \lambda(f(u_{j+1}^{n+1/2}) - f(u_j^{n+1/2})) \\
&\quad + \frac{\Delta t}{2}(R(u_j^{n+1/2}) + R(u_{j+1}^{n+1/2})),
\end{aligned} \tag{3}
$$

where $\lambda = \Delta t/\Delta x$; $u_j'/\Delta x$, $f_j'/\Delta x$ denote first order approximation of space derivatives, and can be obtained by a suitable slope limiter.

2 High Order Schemes

Higher order central schemes for systems of conservation laws (i.e. $R \equiv 0$) can be obtained by using higher order approximation in the reconstruction of the function and in the evaluation of the integral of the flux. The first goal can be obtained by using ENO or WENO reconstructions similar to those used in the context of high order Godunov-type schemes.

High order accuracy in time is obtained by a sufficiently high order quadrature formula for approximating the time integral of the flux. Simpson's rule is sufficient for fourth order schemes. The predictor values at the nodes of the quadrature formula can be efficiently obtained by the use of Runge-Kutta methods and their Natural Continuous Extension.

ENO-based central schemes and Runge-Kutta-NCE for the flux integrals have been introduced in [2]. WENO schemes are even more satisfactory than ENO, since they provide higher (than ENO) order accuracy for the same stencil.

WENO reconstruction is obtained by a convex combination of low order polynomials. In upwind schemes, accuracy up to fifth order is obtained by a combination of three piecewise parabolas. Another (and more important) advantage in using a combination of three parabolas rather than a polynomial of fourth order is that the weights can adjust automatically near a discontinuity of the function, in such a way that only the weight corresponding to the smooth part of the solution will enter in the reconstruction of the function near a discontinuity.

Central WENO schemes have been introduced in [6], where fourth order central schemes have been constructed which make use of a convex combination of three parabolas. A more compact third order scheme is developed in [7], in one and two space dimensions.

3 Treatment of the Source

Central schemes provide a natural way of treating the source term. The source, in fact, can be integrated over the cell, and approximated numerically by a suitable quadrature formula, as in Eq. (3).

The treatment of stiff source is less trivial, since the source has to be treated in some implicit way, in order to guarantee stability. A second order central scheme for the treatment of hyperbolic system with stiff source term is proposed and analyzed in [4].

The scheme can be written as two-stage predictor corrector scheme:

$$
\begin{aligned}
u_j^{n+1/2} &= u_j^n - \frac{\lambda}{2} f_j' + \frac{\Delta t}{2} R(u_j^{n+1/2}) \\
u_j^{n+1/3} &= u_j^n - \frac{\lambda}{3} f_j' + \frac{\Delta t}{3} R(u_j^{n+1/3}) \\
u_{j+1/2}^{n+1} &= \frac{1}{2}(u_j^n + u_{j+1}^n) + \frac{1}{8}(u_j' - u_{j+1}') - \lambda(f(u_{j+1}^{n+1/2}) - f(u_j^{n+1/2})) \\
&\quad + \frac{\Delta t}{8}(3R(u_j^{n+1/3}) + 3R(u_{j+1}^{n+1/3}) + 2R(u_{j+1/2}^{n+1}))
\end{aligned}
$$

It can be shown that the scheme is second order both in the stiff and non stiff limit. Note that both predictors and corrector are implicit in the source term and explicit in the flux. For what concerns the time evolution this is an IMEX (Implicit-Explicit) Runge-Kutta scheme [1].

This scheme is not optimal, because it requires three implicit stages and two explicit stages. A more efficient central scheme, which requires only two implicit stages has been recently presented by Pareschi [9].

4 Future Perspectives

Several directions can be followed to improve and generalize the results obtained so far. We mention here a few research directions in the different topics we have discussed.

High order central schemes. The use of CWENO + RKNCE allows the construction of high order central schemes, which are based on a staggered grid. However, because of the intrinsic limitation of the order of the Natural Continuous Extension, the construction of a fourth order scheme requires five function evaluations (see [2,6]). A more efficient and elegant scheme can be obtained by performing time discretization of the equations first, according to an explicit Runge-Kutta scheme, and performing the space discretization later. In this way a scheme up to fourth order can be obtained with just four function evaluations per cell, which is optimal for explicit fourth order Runge-Kutta schemes [10].

Systems with source. For the construction of high order central schemes for systems with source term it does not seem appropriate to use staggered grids. Non staggered versions of central schemes have been developed. In particular, semidiscrete second and third order central schemes for conservation laws are currently available [3]. Such schemes constitute basically a method of line approach to the problem, and are very attractive, since they allow a great flexibility in the time evolution. Although in some cases they are not as accurate as fully discrete methods (in particular for the propagation of linear discontinuity), they seem to be the most promising approach to the development of high order schemes for systems with source. We remark here that there are other semidiscrete schemes in the literature, which do not require Riemann solvers, and which are therefore very similar, in usage and generality, to semidiscrete central schemes (see, for example, [8]). Once a semidiscrete scheme has been chosen for the hyperbolic part, then the full system can be discretized in time using an IMEX approach. It is desirable to use a scheme in conservation form which is based on the point-wise values of the solution. Such schemes have been widely used in the context of ENO and WENO (see [11]).

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